Auditorium Exercise 01

Differential Equations II for Students of Engineering Sciences
Summer Semester 2024

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Universität Hamburg

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Universität Hamburg
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Overview

General Information

Important Tools

- Fourier Analysis
- Eigenvalue Strategies
- Differential Operators

General Information for Differential Equations II*

- Lecturer: Professor Thomas Schmidt
- Lectures: Mo, 13:15-14:45, A-0.13

- Tutor (English): Jule Schütt
- e-mail: jule.schuett@uni-hamburg.de
- office hour: Mo 10:00-11:00, E4.012

- Auditorium Exercise class: Fr 11:30-13:00, A-1.15
- Exercise groups: Mo 11:30–13:00, A-1.20

- More information and material: math.uni-hamburg.de/teaching

*Everything bi-weekly except lectures

Fourier Analysis

Idea

Having a T-periodic function f (like sin or cos for $T = 2\pi$), then we hope to represent f as uniform convergent series with summands we know well (sin and cos)

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(Hopefully) known tools we need:

- Vector space V
- Scalar product $\langle \cdot, \cdot \rangle$
- Orthonormal basis $b_1, b_2, \ldots$ ($\langle b_i, b_j \rangle = \delta_{i,j}$)

Fourier Analysis

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- Orthonormal basis:

$$u_0(t) := \frac{1}{\sqrt{2}},\ u_k(x) := \cos(k\omega t),\ v_k(t) := \sin(k\omega t),$$

with $k \in \mathbb{N}\,, \omega = \dfrac{2\pi}{T}$.

Indeed, by integration by parts and trigonomic identifications,

$$\frac{2}{T}\int_0^T \sin(k\omega t)\cdot\sin(l\omega t)dt = \begin{cases} 0 & \text{falls } k \neq l, \\ 1 & \text{falls } k = l. \end{cases} \qquad \forall k, l \in \mathbb{N}$$

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$$\frac{2}{T}\int_0^T \frac{1}{\sqrt{2}}\cos(k\omega t)dt = \frac{2}{T}\int_0^T \frac{1}{\sqrt{2}}\sin(k\omega t)dt = 0 \qquad \forall k \in \mathbb{N}\,,$$

$$\frac{2}{T}\int_0^T \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\,dt = 1\,.$$

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So, if $k, l \in \mathbb{N}$, $k \neq l$

$$\begin{aligned}
&\int_0^T 2\sin(k\omega t)\cdot\sin(l\omega t)dt \\
&= \int_0^T \cos((k\omega - l\omega)t) - \cos((k\omega + l\omega)t)\,dt \\
&= \left[\frac{\sin((k\omega - l\omega)t)}{\omega(k-l)} - \frac{\sin((k\omega + l\omega)t)}{\omega(k+l)}\right]_0^T \\
&= \frac{\sin((k-l)\omega T)}{\omega(k-l)} - \frac{\sin((k+l)\omega T)}{\omega(k+l)} \\
&= \frac{\sin((k-l)2\pi)}{\omega(k-l)} + \frac{-\sin((k+l)2\pi)}{\omega(k+l)} \;=\; 0
\end{aligned}$$

For instance,

$$2\sin(\alpha t)\sin(\beta t) \;=\; \cos((\alpha-\beta)t) - \cos((\alpha+\beta)t).$$

and if $k = l$

$$\begin{aligned}
&\int_0^T 2\sin(k\omega t)\cdot\sin(k\omega t)dt \\
&= \int_0^T \cos((k\omega - k\omega)t) - \cos((k\omega + k\omega)t)\, dt \\
&= \int_0^T 1\, dt \;-\; \left[\frac{\sin((2k\omega t)}{2k}\right]_0^T \\
&= T \;-\; \frac{\sin(4k\pi)}{2k} = \; T\,.
\end{aligned}$$

Definition

Let $f : \mathbb{R} \to \mathbb{C}$ be piecewise differentiable and T-periodic, then the **Fourier Series** of f is defined as

$$F_f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \qquad \omega = \frac{2\pi}{T},$$

where we call[†]

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt \overset{\text{if } k\in\mathbb{N}}{=} \langle f, u_k\rangle \qquad k \in \mathbb{N}_0$$

$$b_k = \langle f, v_k\rangle = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt \qquad k \in \mathbb{N}$$

the **Fourier coefficients** of f.

[†] The last equality is only valid for $k \neq 0$ since $a_0/2 = \sqrt{2}\langle f, u_0\rangle$ i.e., we inserted the constant function 1 instead of $1/\sqrt{2}$ in the inner product on the right hand side. One can actually also define $a_0 = \langle f, u_0\rangle$ but then take $a/\sqrt{2}$ instead of $a_0/2$ in the definition of $F_f(t)$.

Convergence

In general ,the series converges to $\frac{1}{2}(f_{-}(t) + f_{+}(t))$ which we denote by $F_f(t) \sim f(t)$.
Hence, if f is continuous at t, then $F_f(t) = f(t)$.

Fourier Analysis: Simplification due to Geometry

If f is **even** $(f(-t) = f(t))$ then

$$b_k = \frac{2}{T} \int_0^T \underbrace{f(t)\sin(k\omega t)}_{\text{odd}}\, dt \;=\; \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(k\omega t)\, dt \;=\; 0$$

and

$$a_k =\; \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \underbrace{f(t)\cos(k\omega t)}_{\text{even}}\, dt \;=\; \frac{4}{T} \int_0^{T/2} f(t)\cos(k\omega t)\, dt$$

If f is **odd** $(f(-t) = -f(t))$ then

$$a_k = 0 \quad \text{und} \quad b_k = \frac{4}{T} \int_0^{T/2} f(t)\sin(k\omega t)\, dt \qquad k \in \mathbb{N}$$

Fourier Analysis: Example 1

Consider

$$f(t) = \begin{cases} 4t & t \in [0, \frac{1}{2}] \\ 4 - 4t & t \in [\frac{1}{2}, 1] \\ 0 & t \in [1, 2] \end{cases}$$

Goal: Fourier series of the 4-periodic **odd** extension of f.

$$T = 4 \qquad \omega = \frac{2\pi}{T} = \frac{\pi}{2} \qquad a_k = 0 \text{ since odd}$$

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$$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega t)\, dt \;=\; \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(k\omega t)\, dt$$

$$= \frac{4}{T} \int_0^{T/2} f(t) \sin(k\omega t)\, dt \;=\; \int_0^2 f(t) \sin(\frac{k\pi}{2} t) dt$$

Fourier Analysis: Example 1

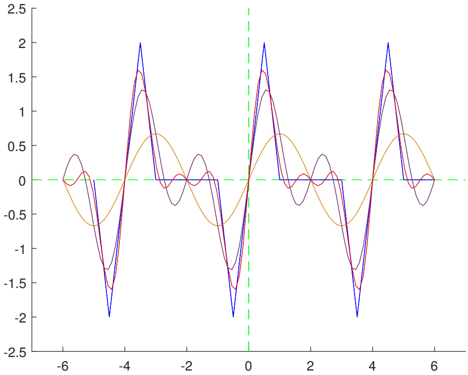
Inserting the definition of f and doing integration by parts gives:

$$
\begin{aligned}
b_k &= \int_0^2 f(t)\sin(\frac{k\pi}{2}t)dt = \left[4t\frac{-\cos\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} 4\frac{-\cos\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}dt \\
&+ \left[(4-4t)\frac{-\cos(\frac{k\pi}{2}t)}{\frac{k\pi}{2}}\right]_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 (-4)\frac{-\cos(\frac{k\pi}{2}t)}{\frac{k\pi}{2}}dt \\
&= \frac{-8}{k\pi}\left(\frac{1}{2}\cos\left(\frac{k\pi}{4}\right)\right) + \frac{8}{k\pi}\left[\frac{\sin\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_0^{\frac{1}{2}} \\
&+ \frac{8}{k\pi}\left(\frac{1}{2}\cos\left(\frac{k\pi}{4}\right)\right) - \frac{8}{k\pi}\left[\frac{\sin\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_{\frac{1}{2}}^1 \\
&= \frac{16}{(k\pi)^2}\sin(\frac{k\pi}{4}) - \frac{16}{(k\pi)^2}\left(\sin(\frac{k\pi}{2}) - \sin(\frac{k\pi}{4})\right) \\
&= \frac{16}{(k\pi)^2}\left(2\sin(\frac{k\pi}{4}) - \sin(\frac{k\pi}{2})\right)
\end{aligned}
$$

Fourier Analysis: Example 1

Since f is continuous and piecewise continuous differentiable,

$$f(t) = \sum_{k=1}^{\infty} b_k \sin(\frac{k\pi}{2} t) = \sum_{k=1}^{\infty} \frac{16}{(k\pi)^2} \left(2\sin(\frac{k\pi}{4}) - \sin(\frac{k\pi}{2}) \right) \sin(\frac{k\pi}{2} t)$$



Fourier Analysis: Example 2

$T = 4$, $g(t) = 3\sin(\frac{3\pi}{2}t)$

Then $\omega = \frac{2\pi}{T} = \frac{\pi}{2}$, and $u_k(t) = \cos(k\frac{\pi}{2}t)$, $v_k(t) = \sin(k\frac{\pi}{2}t)$ for $k \in \mathbb{N}$. In particular, $3v_3(t) = g(t)$. Since u_k and v_k are orthonormal, we conclude

$$a_k = \langle g, u_k \rangle = 3\langle v_3, u_k \rangle = 0$$

for all $k \in \mathbb{N}_0$ and

$$b_k = \langle g, v_k \rangle = 3\langle v_3, v_k \rangle = \begin{cases} 0 & \text{if } k \neq 3 \\ 3 & \text{if } k = 3. \end{cases}$$

Thus, $F_g(t) = g$ for all $t \in \mathbb{R}$.
In particular: If g is a linear combination of sin, cos and constant functions, the computation of the Fourier series simplifies as above.

Eigenvalue Strategies: Reminder & Example: Parameter depending boundary value problem

$$y''(x) + \lambda\, y(x) = 0 \quad y(0) = y(L) = 0 \qquad \text{with } \lambda \in \mathbb{R} \text{ and } L \in \mathbb{R}^+$$

The trivial solution is $y(x) = 0, \forall x \in [0, L]$.

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We want to find out, for which λ there exist non trivial solutions y.
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  - If $\mu_2 = \overline{\mu_1} \notin \mathbb{R}$: $y(x) = c_1 \mathrm{Re}\left(e^{\mu_1 x}\right) + c_2 \mathrm{Im}\left(e^{\mu_1 x}\right)$

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For exercise sheet: Use boundary values for computation of c_1,c_2.

Differential Operators

Definition

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, then the **nabla operator** or **gradient** of f is defined as

$$\nabla f(x) = \nabla f \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_{x_1}(x_1, x_2, \ldots, x_n) \\ f_{x_2}(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_{x_n}(x_1, x_2, \ldots, x_n) \end{pmatrix} = \mathbf{grad} f(x_1, x_2, \ldots, x_n)^T$$

if it exists. The **Laplace operator** Δ is defined as

$$\Delta f(x) = \Delta f(x_1, \ldots, x_n) = \sum_{k=1}^{n} f_{x_k x_k}(x_1, \ldots, x_n)$$

if it exists.

Differential Operators

Definition

Let $v : D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$ be a vector field, then the **divergence** of v is defined as

$$v : \operatorname{div} v(x) = \operatorname{div} v(x_1, \cdots, x_n) = \sum_{k=1}^{n} \frac{\partial v_k}{\partial x_k}(x_1, \cdots, x_n)$$

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Example

If $n = 3$, then $v \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{pmatrix}$,

$$\operatorname{div} v(x, y, z) = \frac{\partial v_1}{\partial x}(x, y, z) + \frac{\partial v_2}{\partial y}(x, y, z) + \frac{\partial v_3}{\partial z}(x, y, z)$$

Interpretation : Volume density of the outward flux (Quelldichte)

Definition

Let $v : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$ be a vector field, then the **rotation** of v is defined as

$$\mathbf{rot}\, v(x,y,z) = \begin{pmatrix} \frac{\partial v_3}{\partial y}(x,\, y,\, z) - \frac{\partial v_2}{\partial z}(x,\, y,\, z) \\ \frac{\partial v_1}{\partial z}(x,\, y,\, z) - \frac{\partial v_3}{\partial x}(x,\, y,\, z) \\ \frac{\partial v_2}{\partial x}(x,\, y,\, z) - \frac{\partial v_1}{\partial y}(x,\, y,\, z) \end{pmatrix}$$

if it exists.

Differential Operators

If we consider plane currents, we can rewrite this in $\mathbb{R}^3$

$$v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1(x,\ y) \\ v_2(x,\ y) \end{pmatrix} \longleftrightarrow \tilde{v}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_1(x,\ y) \\ v_2(x,\ y) \\ 0 \end{pmatrix}$$

For $\tilde{v}$ we obtain the rotation: $(0,\ 0,\ \frac{\partial v_2}{\partial x}(x,\ y) - \frac{\partial v_1}{\partial y}(x,\ y))^T$.

Therefore, we abbreviate $n = 2$:

$$\operatorname{rot} v(x, y) = \frac{\partial v_2}{\partial x}(x,\ y) - \frac{\partial v_1}{\partial y}(x,\ y)$$

Differential Operators: Example 1

Let $v(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} \frac{y}{2} \\ -2x \end{pmatrix}$, $\quad (x, y) \neq (0, 0)$ represent the velocity of a plane current. Then

$$\operatorname{div}(v) = 0 \qquad\qquad \mathbf{rot}(v) = -\frac{5}{2}$$

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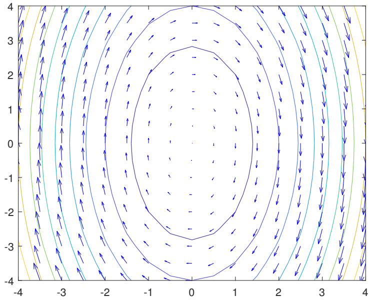

Figure: Example of a vector field with zero divergence (no source/sink behaviour of the flux at any point)

Differential Operators: Example 2 (Combination of operators)

Let $f : D \longrightarrow \mathbb{R}, \quad D \subset \mathbb{R}^3$ be a C^3-function and $v = \nabla f$.

$$\operatorname{div} v(x) = \operatorname{div} \nabla f(x) = \Delta f(x)$$

is well defined and it admits values in $\mathbb{R}$.

$$\nabla \operatorname{div} f(x)$$

is undefined since the divergence is only defined for functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$\nabla \operatorname{div} v(x)$$

is well-defined and it is a vector in $\mathbb{R}^3$ at every point in D.

$$\nabla \,\mathbf{rot} f(x)$$

is undefined since the rotation is only defined for functions from $\mathbb{R}^3$ to $\mathbb{R}^3$.